NONCOMMUTATIVE DIFFERENTIALS AND YANG-MILLS ON PERMUTATION GROUPS S_N

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Abstract. We study noncommutative differential structures on the group of permutations S_N, defined by conjugacy classes. The 2-cycles class defines an exterior algebra Λ_N which is a super analogue of the Fomin-Kirillov algebra $\mathcal{E}_N$ for Schubert calculus on the cohomology of the GL_N flag variety. Noncommutative de Rahm cohomology and moduli of flat connections are computed for $N < 6$. We find that flat connections of submaximal cardinality form a natural representation associated to each conjugacy class, often irreducible, and are analogues of the Dunkl elements in $\mathcal{E}_N$. We also construct Λ_N and $\mathcal{E}_N$ as braided groups in the category of S_N-crossed modules, giving a new approach to the latter that makes sense for all flag varieties.

1. Introduction

In recent years there has been developed a fully systematic approach to the noncommutative differential geometry on (possibly noncommutative) algebras, starting with differential forms on quantum groups[1] and including principal bundles with Hopf algebra fiber, connections and Riemannian structures, etc, see [2] or our companion paper in the present volume for a review. These constructions successfully extend conventional concepts of differential geometry to the q-deformed case such as q-spheres and q-coordinate rings of quantum groups.

However, this constructive noncommutative geometry can also be usefully specialised to finite-dimensional Hopf algebras and from there to finite groups, where differentials and functions noncommute (even though the functions themselves commute). Indeed, one has then a rich 'Lie theory of finite groups' complete with differentials, Yang-Mills theory, metrics and Riemannian structures. If $k(G)$ denotes the functions on the finite group, then the differential structures are defined by exterior algebras of the form $\Omega = k(G).\Lambda$ where Λ is the algebra of left-invariant differential forms. These in turn are determined by conjugacy classes. The case of the symmetric group S_3 of permutations of 3 elements, with its 2-cycle conjugacy class, was fully studied in [2] and [3]. Among other results, it was shown that S_3 has the same noncommutative de Rham cohomology as the quantum group $SL_q(2)$ (or 3-sphere S^3_q in a unitary setting). The moduli space of flat $U(1)$ connections on S_3 is likewise nontrivial and was computed. The goal of the present article is to extend some of these results to higher S_N, with some results for all N and others by explicit computation for $N < 6$.

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In particular, we make a thorough study of the invariant exterior algebra $\Lambda = \Lambda_N$ with the 2-cycles calculus, and explain its close connection with other algebras in mainstream representation theory (in Schubert calculus) and algebraic topology. Our first result is a description of the algebra Λ_N as generated by $\{e_{(ij)}\}$ labelled by 2-cycles with relations

$$e_{(ij)} \wedge e_{(ij)} = 0, \quad e_{(ij)} \wedge e_{(km)} + e_{(km)} \wedge e_{(ij)} = 0$$

$$e_{(ij)} \wedge e_{(jk)} + e_{(jk)} \wedge e_{(ki)} + e_{(ki)} \wedge e_{(ij)} = 0$$

where i, j, k, m are distinct. We consider $e_{(ij)} = e_{(ji)}$ since they are labelled by the same 2-cycle. Our first observation is that Λ_N has identical form to the noncommutative algebra $\mathcal{E}_N$ introduced in [4] with generators $[ij]$ and relations

$$[ij] = -[ji], \quad [ij]^2 = 0, \quad [ij][km] = [km][ij], \quad [ij][jk] + [jk][ki] + [ki][ij] = 0$$

for distinct i, j, k, m. The main difference is that our $e_{(ij)}$ are symmetric and partially anticommute whereas the $[ij]$ are antisymmetric and partially commute. We will show that many of the problems posed in [4] and some of the results there have a direct noncommutative-geometrical meaning in our super version. For example, the algebra $\mathcal{E}_N$ has a subalgebra isomorphic to the cohomology of the flag variety associated to GL_N and among our analogous results we have a subalgebra of flat connections with constant coefficients. These results are in Section 3, with some further metric aspect on Section 5. Moreover, using our methods we obtain several new results about the algebras $\mathcal{E}_N$. These are in Section 6. The first and foremost is our result that the $\mathcal{E}_N$ are braided groups or Hopf algebras in braided categories. We conjecture that as such they are self-dual and show that this unifies and implies several disparate conjectures in [4]. We show that the extended divided-difference operators Δ_{ij} in that paper are indeed the natural braided-differential operators on any braided group, and that the cross product Hopf algebras in [5] are the natural bosonisations. We also prove that if $\mathcal{E}_N$ is finite-dimensional then it has a unique element of top degree. Our approach works for all flag varieties associated to other Lie algebras with Weyl groups beyond S_N.

The reasons for the close relation between Λ_N and $\mathcal{E}_N$ is not known in detail but can be expected to be something like this: the flag variety has a cell decomposition labelled by S_N and its differential geometric invariants should correspond in some sense to the noncommutative discrete geometry of the 'skeleton' of the variety provided by the cell decomposition. One can also consider this novel phenomenon as an extension of Schur-Weyl duality. Let us also note the connnection between flag varieties and the configuration space $C_N(d)$ of ordered N-tuples in $\mathbb{R}^d$ with distinct entries, as emphasized in the recent works of Lehrer, Atiyah and others. Its cohomology ring in the Arnold form can be written as generated by $d-1$-forms E_{ij} labelled by pairs $i \neq j$ in the range $1, \cdots, N$ with relations [6][7]

$$E_{ij} = (-1)^d E_{ji}, \quad E_{ij}E_{km} = (-1)^{d-1}E_{km}E_{ij}, \quad E_{ij}E_{jk} + E_{jk}E_{ki} + E_{ki}E_{ij} = 0$$

for all i, j, k, m not necessarily distinct (to the extent allowed for the labels to be valid). We have rewritten the third relation in the required suggestive form using the first two relations. We see that $H(C_N(d))$ is precisely a graded-commutative version of the algebra Λ_N if d is even and of $\mathcal{E}_N$ extended by dropping the $[ij]^2 = 0$ relation if d is odd. Thus one can say that the noncommutative geometry of S_N and the extended Fomin-Kirillov algebra $\mathcal{E}_N$ together 'quantize' the cohomology of this

configuration space in the sense that some of the graded-commutativity relations are dropped.

In the preliminary Section 2 we recall the basic ingredients of the constructive approach to noncommutative geometry (coming out of quantum groups) that we use. Its relation to other approaches such as [8] is only partly understood, see [2]. In Section 4 of the paper we look at other differential calculi on symmetric groups as defined by other conjugacy classes. To be concrete we look at S_4, S_5 and compute moduli of flat connections with constant coefficients. The result suggests the (incomplete) beginnings of an approach to construct an irreducible representation associated to each conjugacy class by noncommutative-geometrical means and in a manner that would make sense in principle for any finite group G. Since all of the geometry is G-equivariant there is plenty of scope to associate representations; here we explore one such method and tabulate the results.

2. Preliminaries on noncommutative differentials

Noncommutative differential geometry works over a general unital (say) algebra A. The main idea is to define the differential structure by specifying an $A-A$-bimodule Ω^1 of '1-forms' equipped with an exterior derivative $\mathrm{d}: A \to \Omega^1$ obeying the Leibniz rule. When A is a Hopf algebra there is a natural notion of Ω^1 bicovariant [1] and in this case it can be shown that $\Omega^1 = A.\Lambda^1$ (a free left A-module) where Λ^1 is the space of left-invariant 1-forms. This space has the natural structure of a right A-crossed module (in the case of A finite-dimensional it means a right module over the right quantum double of A) and as a result a braiding operator $\Psi : \Lambda^1 \otimes \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$ obeying the Yang-Baxter equations. This can be used to define the wedge product between invariant 1-forms in such a way that they 'skew-commute' with respect to Ψ. The naive prescription is a quadratic algebra Λ_{quad} but there is also a more sophisticated Woronowicz prescription Λ_w; in both cases the exterior algebra Ω is defined as freely generated by these over A. Here

$$\Lambda_{quad} = T\Lambda^1/\ker(\mathrm{id} - \Psi), \quad \Lambda_w = T\Lambda^1/\oplus_n \ker A_n \tag{1}$$

as quotients of the tensor algebra, where the Woronowicz antisymmetrizer is

$$A_n = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \Psi_{i_1} \cdots \Psi_{i_{l(\sigma)}} : (\Lambda^1)^{\otimes n} \to (\Lambda^1)^{\otimes n}. \tag{2}$$

Here $\Psi_i \equiv \Psi_{i,i+1}$ denotes Ψ acting in the $i, i+1$ place and $\sigma = s_{i_1} \cdots s_{i_{l(\sigma)}}$ is a reduced expression in terms of simple reflections. There is also an operator $\mathrm{d} : \Lambda^1 \to \Lambda^2$ which extends to the entire exterior algebra with $\mathrm{d}^2 = 0$ and defines the noncommutative de Rham cohomology as closed forms modulo exact.

Proposition 2.1. [9] $A_n = [n; -\Psi]!$ *where*

$$[n; -\Psi] = \mathrm{id} - \Psi_{12} + \Psi_{12}\Psi_{23} + \cdots + (-1)^{n-1}\Psi_{12} \cdots \Psi_{n-1,n}$$

are the braided integer matrices and $[n; -\Psi]! = (\mathrm{id} \otimes [n-1; -\Psi]!)[n; -\Psi]$.

This is a practical method to compute the A_n, which we will use. It comes from the author's theory of braided binomials (or sometimes called braided shuffles) introduced in [12]. See also the later works [10][11]. For example,

$$\begin{aligned}[3; -\Psi]! &= (\mathrm{id} \otimes [2; -\Psi])[3; -\Psi] = (\mathrm{id} - \Psi_{23})(\mathrm{id} - \Psi_{12} + \Psi_{12}\Psi_{23}) \\ &= \mathrm{id} - \Psi_{12} - \Psi_{23} + \Psi_{12}\Psi_{23} + \Psi_{23}\Psi_{12} - \Psi_{23}\Psi_{12}\Psi_{23} = A_3.\end{aligned}$$

For other formulae it is enough for our purposes to specialise directly to finite sets and finite groups. We work over a field k of characteristic zero. Let $A = k(\Sigma)$ a finite set. Then the differential structures are easily seen from the axioms to correspond to subsets $E \subset \Sigma \times \Sigma - \mathrm{diag}$ of 'allowed directions'. Thus

$$\Omega^1 = \mathrm{span}\{\delta_x \otimes \delta_y |\ (x,y) \in E\}, \quad \mathrm{d}f = \sum_{(x,y)\in E} (f(y) - f(x))\delta_x \otimes \delta_y \tag{3}$$

where δ_x is the Kronecker delta-function. Note that $\delta_x \otimes \delta_y = \delta_x \mathrm{d}\delta_y$ for all $(x,y) \in E$. This result for finite sets is common to all approaches to noncommutative geometry, e.g. in [8]. If $\Sigma = G$ is a finite group then a natural choice of E is given by

$$E = \{(x,y) \in G \times G |\ x^{-1}y \in \mathcal{C}\} \tag{4}$$

for any subset $\mathcal{C}$ not containing the group identity e. Such a calculus is manifestly invariant under translation by G and all covariant differential calculi are of this form. Bicovariant ones (as above) are given precisely by those $\mathcal{C}$ which are Ad-stable, so that E is invariant from both sides. The 'simple' such differential structures (with no proper quotient) are classified precisely by the nontrivial conjugacy classes. They take the form of a free left $k(G)$-module

$$\Omega^1 = k(G) \cdot \mathrm{span}\{e_a |\ a \in \mathcal{C}\}, \quad \mathrm{d}f = \sum_{a\in\mathcal{C}} (R_a(f) - f)e_a, \quad e_a f = R_a(f)e_a \tag{5}$$

where $R_a(f)(g) = f(ga)$ denotes right translation and, explicitly, $e_a = \sum_{g\in G} \delta_g \mathrm{d}\delta_{ga}$. Such formulae follow at once from Woronowicz's paper as a special case. An early study of this case, in the physics literature, is in [13].

Moreover, in the case of a finite group G, a right $k(G)$-crossed module is the same thing (by evaluation) as a left G-crossed module in the sense of Whitehead, i.e. a G-graded G-module with the degree map $|\ |$ from the module to kG being equivariant (where G acts on kG by Ad), see[14]. The particular crossed module structure on $\Lambda^1 = k\mathcal{C}$ and induced braiding are

$$|e_a| = a, \quad g.e_a = e_{gag^{-1}}, \quad \Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a. \tag{6}$$

Proposition 2.2. [2] *For each $g \in G$, consider the set $\mathcal{C} \cap g\mathcal{C}^{-1}$. This has an automorphism $\sigma(a) = a^{-1}g$ corresponding to the braiding under the decomposition $k\mathcal{C} \otimes k\mathcal{C} = \sum_g k(\mathcal{C} \cap g\mathcal{C}^{-1})$. Hence if $V_g = (k\mathcal{C} \cap g\mathcal{C}^{-1})^\sigma$ (the fixed subspace) has basis $\{\lambda^{(g)\alpha}\}$, the full set of relations of Λ_{quad} are*

$$\forall g \in G: \qquad \sum_{a,b\in\mathcal{C},\, ab=g} \lambda_a^{(g)\alpha} e_a e_b = 0.$$

These are also the relations of Ω_{quad} over $k(G)$. Meanwhile, the exterior derivative is provided by

$$\mathrm{d}e_a = \theta e_a + e_a \theta, \quad \theta = \sum_{a\in\mathcal{C}} e_a. \tag{7}$$

It follows that d is given in all degrees by graded-commutation with the 1-form θ. It is easy to see that it obeys $\theta^2 = 0$ and $\mathrm{d}\theta = 0$ and that θ is never exact (so the noncommutative de Rham cohomology H^1 always contains the class of θ).

3. 2-Cycle differential structure on S_N

It is straightforward to compute the quadratic exterior algebra for $G = S_N$ from the above definitions. We are particularly interested in the invariant differential forms since these generate the full structure over $k(G)$. In this section, we take the differential structure defined by the conjugacy class $\mathcal{C}$ consisting of 2-cycles described as unordered pairs (ij) for distinct $i, j \in \{1, \cdots, N\}$.

Proposition 3.1. *The quadratic exterior algebra $\Lambda_N \equiv \Lambda_{quad}(S_N)$ for the 2-cycles class is the algebra with generators $\{e_{(ij)}\}$ and relations*

$$(i) \quad e_{(ij)}^2 = 0, \quad (ii) \quad e_{(ij)}e_{(km)} + e_{(km)}e_{(ij)} = 0$$
$$(iii) \qquad e_{(ij)}e_{(jk)} + e_{(jk)}e_{(ik)} + e_{(ik)}e_{(ij)} = 0$$

where i, j, k, m are distinct.

Proof. There are three kinds of elements $g \in G$ for which $\mathcal{C} \cap g\mathcal{C}^{-1}$ is not empty. These are (i) $g = e$, in which case σ is trivial and $V_e = k\mathcal{C}$. This gives the relations (i) stated; (ii) $g = (ij)(km)$ where i, j, k, m are disjoint. In this case $\mathcal{C} \cap g\mathcal{C}^{-1}$ has two elements (ij) and (km), interchanged by σ. The basis of $V_{(ij)(km)}$ is 1-dimensional, namely $(ij) + (km)$ and this gives the relation (ii) stated; (iii) The element $g = (ij)(jk)$ where i, j, k are disjoint. Here $\mathcal{C} \cap g\mathcal{C}^{-1}$ has 3 elements $(ij), (jk), (ik)$ cyclically rotated by σ. The invariant subspace is 1-dimensional with basis $(ij) + (jk) + (ik)$ giving the relation (iii). $\diamond$

We note that

$$\dim(\Lambda_N^1) = \binom{N}{2}, \quad \dim(\Lambda_N^2) = \frac{N(N-1)(N-2)(3N+7)}{24}$$

which are the same dimensions as for the algebra $\mathcal{E}_N$ in [4]. The first of these is the 'cotangent dimension' of the noncommutative manifold structure on S_N. It is more or less clear from the form of the two algebras that their dimensions coincide in all degrees (and for $N = 3$ they are actually isomorphic). We have computed these dimensions for the exterior algebra for $N < 6$ using the explicit form of the braiding Ψ defining the algebra, and they indeed coincide with the corresponding dimensions for $\mathcal{E}_N$ listed in [4]. These data are listed in Table 1 with the compact form of the Hilbert series taken from [4] (for S_4, S_5 only the low degrees have been explicitly verified by us). Also of interest is the top degree d in the last column. From our noncommutative geometry point of view this is the 'volume dimension' of the noncommutative manifold structure where the top form plays the role of the volume form. Note that the cotangent dimension and volume dimension need not coincide even though they would do so in classical geometry. Also note (thanks to a comment by R. Marsh) that these volume dimensions are exactly the number of indecomposable modules of the preprojective algebra of type SL_N. The latter is a quotient of the path algebra of the doubled quiver of the associated oriented Dynkin diagram and a module means an assignment of 'parallel transport' operators to arrows of the quiver, i.e. some kind of 'connection'. A classic theorem of Lusztig-Kashiwara-Saito states that there is a 1-1 correspondence between the irreducible components of its module variety with fixed dimension vector and the canonical basis elements of the same weight. Since the representation theory for the next preprojective algebra in the series is tame but infinite, we therefore expect Λ_6 and higher to be infinite-dimensional, and similarly for $\mathcal{E}_6$ and higher.

dim	Ω^0	Ω^1	Ω^2	Ω^3	Ω^4	Hilbert polynomial(q)	Top degree
S_2	1	1				$[2]_q$	1
S_3	1	3	4	3	1	$[2]_q^2[3]_q$	4
S_4	1	6	19	42	71	$[2]_q^2[3]_q^2[4]_q^2$	12
S_5	1	10	55	220	711	$[4]_q^4[5]_q^2[6]_q^4$	40

TABLE 1. Dimensions and Hilbert polynomial for the exterior algebras Λ_{quad} for $N < 6$ as for $\mathcal{E}_N$ in [4]. Here $[n]_q = (q^n-1)/(q-1)$.

The infinite-dimensionality or not of $\mathcal{E}_6$ has been posed in [4], where it was conjectured that if finite dimensional then the top form should be unique (we will prove this in Section 6) and that the Hilbert series should have a symmetric increasing and decreasing form. Without proving this second conjecture here, let us outline a noncommutative-geometric strategy for its proof. Namely, the Woronowicz quotient Λ_w by its very construction will be nondegenerately paired with a similar algebra Λ_w^* of 'skew tensor fields' (see the Appendix). Moreover, if finite dimensional, and in the presence of a nondegenerate metric (see in Section 5) we then expect Hodge $*$ isomorphisms $\Lambda_w^m \to \Lambda_w^{d-m}$ and ultimately an increasing-decreasing symmetric form of the Hilbert series for Λ_w as familiar in differential geometry. All of this was concretely demonstrated for S_3 in [3]. The main ingredient missing then is that Λ_N is the quadratic quotient whereas the Woronowicz one could in principle be a quotient of that. The same strategy and considerations apply to $\mathcal{E}_N$. As a step we have,

Theorem 3.2. *For the 2-cycle differential calculus on S_N, $\Lambda_w = \Lambda_{quad}$ in degree < 4 (we conjecture this for all degrees).*

Proof. We will decompose the space $k\mathcal{C}\otimes k\mathcal{C}\otimes k\mathcal{C} = V_3\oplus V_2\oplus V_1\oplus V_0$ where each V_i is stable under the braiding operators Ψ_{12}, Ψ_{23}. Since A_3 can be factorised either through $\mathrm{id}-\Psi_{12}$ or $\mathrm{id}-\Psi_{23}$, its kernel contains that of these operators. So it suffices to show on each V_i that the dimension of the kernel of A_3 equals the dimension of the sum of the kernels of $\mathrm{id}-\Psi_{12}, \mathrm{id}-\Psi_{23}$. We say $a \sim b$ if the 2-cycles a, b have exactly one entry in common and $a \perp b$ if disjoint. We then decompose $\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ as follows. For V_0 we take triples (a, b, c) which are pairwise either $\perp$ or equal, but not all three equal. Here the braiding is trivial. For V_1 we take triples where two pairs are mutually $\perp$ and one is $\sim$. It suffices to let the totally disjoint element be fixed, say (45) and the others to have entries taken from a fixed set, say $\{1, 2, 3\}$ (i.e. V_1 is a direct sum of stable subspaces spanned by basis triples with these properties fixed). On such a 9-dimensional space one may compute Ψ_{12}, Ψ_{23} explicitly and verify the required kernel dimensions (for A_3 it is 7). For V_2 we take triples (a, b, c) where two pairs are $\sim$ and one is $\perp$, or where all three pairs are $\sim$ through the same entry occurring in all three 2-cycles. This time it suffices to take entries from $\{1, 2, 3, 4\}$, say, and verify the kernels on such a 16-dimensional subspace (for A_3 it is 11 dimensional). For V_3 we take triples which are pairwise either $\sim$ or $=$, excluding the special subcase of three $\sim$ used in V_2. Here it suffices to take entries from $\{1, 2, 3\}$ and the braidings become as for S_3, where the result is known from [3]. The remaining type of triple, where there is one pair $\perp$, one $=$ and one $\sim$, is not possible. $\diamond$

The absence of additional cubic relations strongly suggests that the Woronowicz exterior algebra on S_N coincides with the quadratic one in all degrees (this is known for $N = 2, 3$ by direct computation). In view of the above theorem, we continue to work throughout with the quadratic exterior algebra. On the other hand, it should be stressed that we expect $\Lambda_w = \Lambda_{quad}$ to be a special feature of S_N. The evidence for this is that one may expect a kind of 'Schur-Weyl duality' between the noncommutative geometry of the finite group on one side and that of the classical or quantum group on the other. And on the quantum group side it is known that the Woronowicz exterior algebra of $SL_q(N)$ coincides with the quadratic one for generic q, but not for the other classical families. Therefore for other than the SL_N series we would expect to need to work with the nonquadratic Λ_w and likewise propose a corresponding nonquadratic antisymmetric version generalising the $\mathcal{E}_N$.

Next, for any differential graded algebra we define cohomology as usual, namely closed forms modulo exact. It is easy to see that $H^0(S_N) = k.1$ for all N.

Proposition 3.3. *The first noncommutative de Rham cohomology of S_N at least for $N < 6$ with the 2-cycle differential structure is*

$$H^1(S_N) = k.\theta$$

Proof. This is done by direct computation of the dimension of the kernel of d, along the same lines as in [3], after which the result follows. We expect that in fact $H^1(S_N) = k.\theta$ for all N, but the general proof requires some elaboration. $\diamond$

It follows from Poincaré duality that $H^2(S_3) = 0$ and $H^3(S_3) = k$, $H^4(S_3) = k$ as computed explicitly in [3], which is the same as for $SL_q(2)$ and gives some small evidence for the Schur-Weyl duality mentioned above (up to a shift or mismatch in the rank). Next, beyond the cohomology H^1 is a nonlinear variant which can be called '$U(1)$ Yang-Mills theory'. Here a connection or gauge field is again a 1-form $\alpha \in \Omega^1$. But rather than modulo exact 1-forms we are interested in working modulo the gauge transformation

$$\alpha \mapsto u\alpha u^{-1} + u\mathrm{d}u^{-1}$$

for invertible u in our coordinate algebra. The covariant curvature of a connection is

$$F(\alpha) = \mathrm{d}\alpha + \alpha^2$$

and transforms by conjugation. This is like nonAbelian gauge theory but is nonlinear even for the $U(1)$ case because the differential calculus is noncommutative.

Proposition 3.4. *For the 2-cycle differential calculus on S_N,*

$$\alpha_i = -\theta_i, \quad \theta_i = \sum_{j \neq i} e_{(ij)}$$

are flat connections with constant coefficients. The 1-forms θ_i obey $\theta_i\theta_j + \theta_j\theta_i = 0$ for $i \neq j$.

Proof. We first check the anticommutativity for $i \neq j$. In the sum

$$\theta_i\theta_j + \theta_j\theta_i = \sum_{k \neq i, l \neq j} e_{(ik)}e_{(jl)} + e_{(jl)}e_{(ik)}$$

only the cases where i, j, k, l are not distinct contribute due to relation (ii) in Proposition 3.1. Likewise the terms where $(ik) = (jl)$ do not contribute by (i). There are

three remaining and mutually exclusive cases: $k=l$, or $k=j$ or $l=i$. Relabelling the summation variable k in each case we have,

$$\theta_i\theta_j+\theta_j\theta_i=\sum_{k\neq i,j}e_{(ik)}e_{(jk)}+e_{(jk)}e_{(ik)}+e_{(ij)}e_{(jk)}+e_{(jk)}e_{(ij)}+e_{(ik)}e_{(ji)}+e_{(ji)}e_{(ik)}=0$$

by relation (iii). Next, we note that $\sum_i\theta_i=2\theta$. Hence, $\mathrm{d}\alpha_i=\theta\alpha_i+\alpha_i\theta=-\frac{1}{2}\sum_k\alpha_k\alpha_i+\alpha_i\alpha_k=-\alpha_i^2$ as required. $\diamond$

In the algebra $\mathcal{E}_N$ the similar elements

$$\theta_i=\sum_{i<j}[ij]-\sum_{j<i}[ji]=\sum_{j\neq i}[ij] \tag{8}$$

form a commutative subalgebra isomorphic to the cohomology of the flag variety[4]. In our case we see that they anticommute rather than commute. Also, while the elementary symmetric polynomials of the θ_i in $\mathcal{E}_N$ vanish, we have in Λ_N

$$\sum_i\theta_i=2\theta,\quad \sum_i\theta_i^2=0$$

as above. On the other hand, in [4] the generators θ_i are motivated from Dunkl operators on the cohomology of the flag variety but in our case they have a direct noncommutative geometrical interpretation as flat connections. We will see in the next section that they are precisely the flat connections with constant coefficients of minimal support.

4. General differentials and flat connections up to S_5

So far we have studied only one natural conjugacy class. However, our approach associates a similar exterior algebra for any nontrivial conjugacy class in a finite group G. Moreover, since our constructions are G-invariant, we will obtain 'geometrically' plenty of G-modules naturally associated to the conjugacy class. The cohomology does not tend to be a very interesting representation but the moduli of flat connections turns out to be more nontrivial and we will see that for S_N it does yield interesting irreducible modules. We begin with some remarks for general finite groups G equipped with a choice of nontrivial conjugacy class.

First of all, the space of connections is an affine space. We take as 'reference' the form $-\theta$. Then one may easily see that the differences $\phi\equiv\alpha+\theta$ transform covariantly as

$$\phi=\sum_a\phi^ae_a\mapsto u\phi u^{-1}=\sum_a\frac{u}{R_a(u)}\phi^ae_a. \tag{9}$$

Moreover, the curvature of α is

$$F(\alpha)=\mathrm{d}\alpha+\alpha^2=\mathrm{d}(\phi-\theta)+(\phi-\theta)^2=\phi^2 \tag{10}$$

in view of the properties of θ.

Lemma 4.1. *Let $\alpha\in\Omega^1(G)$ be a connection. We define its 'cardinality' to be the number of nonzero components of $\alpha+\theta$ in the basis $\{e_a\}$. This is gauge-invariant and stratifies the moduli of connections.*

Proof. A gauge transformation u is invertible hence the support of each component ϕ^a is gauge-invariant under the transformation shown above. In particular, the number of ϕ^a with nontrivial support is invariant. $\diamond$

We are particularly interested in invariant forms Λ_{quad} and hence connections with constant coefficients ϕ^a (otherwise we have more refined gauge-invariant support data, namely an integer-valued vector whose entries are the cardinality of the support of each ϕ^a). To simplify the problem further we restrict to constant coefficients in $\{0,1\}$. We can project any connection to such a $\{0,1\}$-connection by replacing non-zero ϕ^a by 1, so this limited class of connections gives useful information about any connection.

Proposition 4.2. *Flat connections with constant coefficients in* $\{0,1\}$ *are in correspondence with subsets* $X\subseteq\mathcal{C}$ *such that*

$$\mathrm{Ad}_x(X)=X,\quad \forall x\in X.$$

The intersection of such subsets provides a product in the moduli of such flat connections that non-strictly lowers cardinality. The stratum F_n *of subsets of a given cardinality* n *is* G*-invariant under* Ad.

Proof. The correspondence between $\{0,1\}$-connections and subsets is via the support of the components ϕ^a regarded as a function of $a\in\mathcal{C}$ (so the cardinality of the connection is that of the subset.) We have to solve the equation $\phi^2=0$. But the relations in Λ_{quad} are defined by the braiding Ψ and hence this equation is

$$\Psi(\phi\otimes\phi)=\phi\otimes\phi.$$

Using the form of Ψ this is

$$0=\sum_{a,b\in\mathcal{C}}\phi^a\phi^b(e_{aba^{-1}}\otimes e_a-e_a\otimes e_b)$$

or

$$\phi^a(\phi^{a^{-1}ba}-\phi^b)=0\quad\forall a,b\in\mathcal{C}.$$

This translates into the characterisation shown. On the other hand, this characterisation is clearly closed under intersection. Finally, if X is such a subset then $Y=\mathrm{Ad}_g(X)$ is another such subset because if $y=gxg^{-1}$ and $z=gwg^{-1}$ for $x,w\in X$ then $\mathrm{Ad}_y(z)=(gxg^{-1})gwg^{-1}(gx^{-1}g^{-1})=g(\mathrm{Ad}_x(w)g^{-1}$ is in Y. $\diamond$

Over $\{0,1\}$ the stratum of top cardinality has one point, $X=\mathcal{C}$, which corresponds to $\alpha=0$ or $\phi=\theta$. The stratum of zero cardinality likewise has one point, $X=\emptyset$ corresponding to $\alpha=-\theta$ or $\phi=0$, and the stratum with cardinality 1 can be identified with $\mathcal{C}$, with $\alpha=e_a-\theta$ or $\phi=e_a$ for $a\in\mathcal{C}$. In between these, the spans kF_n are natural sources of permutation G-modules. More precisely one typically has $\theta\in kF_n$ and we look at the module

$$V_n=kF_n/k\theta. \tag{11}$$

For example, the 'submaximal stratum' (the one below the top one) associates a representation to a conjugacy class, i.e. is an example of an 'orbit method' for finite groups. Like the usual orbit method for Lie groups, it does not always yield an irreducible representation, but does sometimes. It should be stressed that this is only one example of the use of our geometrical methods to define representations and we present it only as a first idea towards a more convincing orbit method.

S_3	$\lvert\mathcal{C}\rvert$	Solutions/k	Repn $kF_n/k\theta$	Specht
(12)	3	$F_3=\{\theta\}$		fund
		$F_1=\{e_{23},e_{13},e_{12}\}$	fund	
(123)	2	$F_2=\{\cdot e_{123}+\cdot e_{132}\}$		sign
		$F_1=\{e_{123},e_{321}\}$	sign	

TABLE 2. Flat connections with constant coefficients on S_3 for each conjugacy class, listed by cardinality. $\cdot$ denotes independent nonzero multiples are allowed.

We now use the permutation groups S_N to explore these ideas concretely. For S_3 the full moduli of (unitary) flat connections has already been found in [3] for the 2-cycles class, while the other class is more trivial. The exterior algebra in the second case is $\Lambda_w=\Lambda_{quad}=k\langle e_{123},e_{132}\rangle$ modulo the relations

$$e_{123}^2=0,\quad e_{132}^2=0,\quad e_{123}e_{132}+e_{132}e_{123}=0$$

(a Grassmann 2-plane). For brevity, we suppress the brackets, so $e_{123}\equiv e_{(123)}$, etc.

Proposition 4.3. *The set of flat connections with constant coefficients for S_3,S_4 with their various conjugacy classes are as shown in Tables 2,3. For each stratum F_n of cardinality n, we list the corresponding ϕ up to an overall scale. The associated representations V_n turn out to be irreducible.*

Proof. This is done by direct computation. Note that the entries ϕ of a discrete stratum each define a line of flat connections $\alpha=\lambda\phi-\theta$ for a parameter λ. The entries $\cdot e_{123}+\cdot e_{132}$, etc., define a plane of connections $\alpha=\lambda e_{123}+\mu e_{132}$. As above, we omit the brackets on the cycles labelling the e_a, for example $e_{12,34}$ denotes $e_{(12)(34)}$. For the V_n we enumerate the flat connections in the stratum with coefficients $\{0,1\}$. The resulting representation is then recognised using character theory. Representations are labelled by dimension and by $^-$ if the character at (12) is negative. The fundamental representation of S_4 means the standard 3-dimensional one. $\diamond$

For comparison, the tables also list the standard Specht module of the Young tableau of conjugate shape to that of the conjugacy class. We see that our 'orbit method' produces comparable (although different) answers. As was to be expected, we do not obtain all irreducibles from consideration of $\{0,1\}$ connections alone. Similarly for the S_5 case:

Proposition 4.4. *The set of flat connections with constant coefficients $\{0,1\}$ for S_5 with its various conjugacy classes are as shown in Table 4, organised by stratum F_n of cardinality n. The submaximal strata are shown in detail as well as their associated representations V_n.*

Proof. These results have been obtained with GAP to compute Ad tables followed by MATHEMATICA running for several days to enumerate the flat connections. In the tables θ_i denotes a sum over the relevant size cycles containing i, extending our previous notation. $\diamond$

We see that for low N the V_n tend to be irreducible, but they are not always. Notably, the (123)(45) conjugacy class for S_5 has a 9-dimensional representation

S_4	$\lvert\mathcal{C}\rvert$	Solutions/k	Repn $kF_n/k\theta$	Specht
(12)	6	$F_6=\{\theta\}$ $F_3=\{\theta-\theta_i\}$ $F_2=\{\cdot e_{14}+\cdot e_{23},\cdot e_{13}+\cdot e_{24},\cdot e_{12}+\cdot e_{34}\}$ $F_1=\{e_a\}$	 fund 2 	fund
(12)(34)	3	$F_3=\{\cdot e_{12,34}+\cdot e_{13,24}+\cdot e_{14,23}\}$ $F_2=\{\cdot e_{12,34}+\cdot e_{13,24},\cdot e_{13,24}+\cdot e_{14,23},\cdot e_{12,34}+\cdot e_{14,23}\}$ $F_1=\{e_a\}$	 2 	2
(123)	8	$F_8=\{\cdot(e_{123}+e_{142}+e_{134}+e_{243})+\cdot(e_{132}+e_{124}+e_{143}+e_{234})\}$ $F_4=\left\{\begin{array}{c} e_{123}+e_{142}+e_{134}+e_{243},\\ e_{132}+e_{124}+e_{143}+e_{234}\end{array}\right\}$ $F_2=\{\cdot e_{123}+\cdot e_{132},\cdot e_{142}+\cdot e_{124},\cdot e_{134}+\cdot e_{143},\cdot e_{243}+\cdot e_{234}\}$ $F_1=\{e_a\}$	 sign fund 	$\overline{\text{fund}}$
(1234)	6	$F_6=\{\theta\}$ $F_2=\{\cdot e_{1234}+\cdot e_{1432},\cdot e_{1243}+\cdot e_{1342},\cdot e_{1324}+\cdot e_{1423}\}$ $F_1=\{e_a\}$	 2 	sign

TABLE 3. Flat connections with constant coefficients on S_4 for each conjugacy class, listed by cardinality. $\cdot$ denotes independent nonzero multiples are allowed.

associated to the submaximal stratum. A possible refinement would be to consider only the 'discrete series' i.e. flat connections where ϕ of a fixed normalisaiton is not deformable. For S_4 this means from Table 3 the strata F_3 for the 2-cycles class and F_4 for 3-cycles. A different problem is that one does not get all irreducibles in this way (since one only gets permutation modules). To go beyond this one could consider the full moduli of flat connections including the non-discrete series but with other constraints. For example, one could consider connections with values in $\{-1,0,1\}$, or one could introduce further geometric ideas such as 'polarizations' to our noncommutative setting.

Finally, each of our conjugacy classes on S_N has its associated quadratic algebra Λ_{quad} of interest in its own right. We give just one example.

Proposition 4.5. *For S_4 with its 3-cycle conjugacy class the exterior algebra Λ_{quad} has relations*

$$e_{xyz}^2=0,\quad e_{xyz}e_{xzy}+e_{xzy}e_{xyz}=0,$$
$$e_{123}e_{134}+e_{134}e_{142}+e_{142}e_{123}=0,\quad e_{123}e_{243}+e_{243}e_{134}+e_{134}e_{123}=0,$$
$$e_{134}e_{243}+e_{243}e_{142}+e_{142}e_{134}=0,\quad e_{123}e_{142}+e_{142}e_{243}+e_{243}e_{123}=0,$$
$$e_{123}e_{124}+e_{124}e_{134}+e_{134}e_{234}+e_{234}e_{123}=0,$$
$$e_{123}e_{143}+e_{143}e_{243}+e_{243}e_{124}+e_{124}e_{123}=0,$$
$$e_{123}e_{234}+e_{234}e_{142}+e_{142}e_{143}+e_{143}e_{123}=0$$

and their seven conjugate-transposes (i.e. replacing e_{xyz} by e_{xzy} and reversing products).

Proof. Direct computation of the kernel of $\mathrm{id}-\Psi$ using GAP and MATHEMATICA. We omit the brackets around the 3-cycle labels (as above). $\diamond$

S_5	$\lvert\mathcal{C}\rvert$	Solutions/$\mathbb{Z}_2$	Repn $kF_n/k\theta$
(12)	10	$F_{10} = \{\theta\}$ $F_6 = \{\theta - \theta_i\}$ $\lvert F_4\rvert = 10,\ \lvert F_3\rvert = 10,\ \lvert F_2\rvert = 15$ $F_1 = \{e_a\}$	fund
(12)(34)	15	$F_{15} = \{\theta\}$ $F_5 = \left\{\begin{array}{c} e_{14,23} + e_{12,35} + e_{13,45} + e_{25,34} + e_{15,24}, \\ e_{14,23} + e_{13,25} + e_{24,35} + e_{15,34} + e_{12,45}, \\ e_{13,24} + e_{12,35} + e_{24,35} + e_{12,45} + e_{15,24}, \\ e_{13,24} + e_{15,23} + e_{25,34} + e_{12,45} + e_{14,35}, \\ e_{12,34} + e_{13,25} + e_{23,45} + e_{15,24} + e_{14,35}, \\ e_{12,34} + e_{15,23} + e_{24,35} + e_{14,25} + e_{13,45} \end{array}\right\}$ $\lvert F_3\rvert = 15,\ \lvert F_2\rvert = 15$ $F_1 = \{e_a\}$	$\bar{5}$
(123)	20	$F_{20} = \{\theta\}$ $F_8 = \{\theta - \theta_i\}$ $\lvert F_4\rvert = 10,\ \lvert F_2\rvert = 10$ $F_1 = \{e_a\}$	fund
(123)(45)	20	$F_{20} = \{\theta\}$ $F_2 = \{e_{xyz,12} + e_{xzy,12}, \cdots (\text{all } 2-\text{cycles; xyz complementary})\}$ $F_1 = \{e_a\}$	fund $\oplus$ 5
(1234)	30	$F_{30} = \{\theta\}$ $F_{10} = \left\{\begin{array}{l} e_{1234} + e_{1523} + e_{2435} + e_{2534} + e_{1245} \\ \quad + e_{1542} + e_{1354} + e_{1453} + e_{1432} + e_{1325}, \\ e_{1234} + e_{1253} + e_{2453} + e_{2354} + e_{1524} \\ \quad + e_{1425} + e_{1345} + e_{1543} + e_{1432} + e_{1352}, \\ e_{1234} + e_{1253} + e_{2453} + e_{2354} + e_{1524} \\ \quad + e_{1425} + e_{1345} + e_{1543} + e_{1432} + e_{1352}, \\ e_{1523} + e_{2453} + e_{2354} + e_{1243} + e_{1452} \\ \quad + e_{1254} + e_{1534} + e_{1435} + e_{1342} + e_{1325}, \\ e_{1532} + e_{2435} + e_{2534} + e_{1452} + e_{1254} \\ \quad + e_{1345} + e_{1324} + e_{1543} + e_{1423} + e_{1235}, \\ e_{1253} + e_{2345} + e_{2543} + e_{1245} + e_{1542} \\ \quad + e_{1534} + e_{1435} + e_{1324} + e_{1423} + e_{1352} \end{array}\right\}$ $F_6 = \{\theta - \theta_i\}$ $\lvert F_5\rvert = 12,\ \lvert F_2\rvert = 15$ $F_1 = \{e_a\}$	$\bar{5}$
(12345)	24	$F_{24} = \{\theta\}$ $F_{12} = \left\{\begin{array}{c} e_{12345} + e_{12453} + e_{12534} + \cdots (\text{sum even}), \\ e_{12354} + e_{12435} + e_{12543} + \cdots (\text{sum odd}) \end{array}\right\}$ $\lvert F_4\rvert = 6,\ \lvert F_3\rvert = 24,\ \lvert F_2\rvert = 36$ $F_1 = \{e_a\}$	sign

Table 4. Flat connections with constant coefficients in $\{0, 1\}$ on S_5 for each conjugacy class. The submaximal strata are listed in detail, as well as the associated representation.

We conclude with one general result pertaining to the above ideas.

Proposition 4.6. *For the 2-cycles conjugacy class on S_N, the flat connections with constant coefficients in $\{0,1\}$ of submaximal cardinality are precisely the $\alpha_i = -\theta_i$ in Proposition 3.4. The associated module is the fundamental representation of S_N.*

Proof. The α_i correspond to $\phi_i = \theta - \theta_i$ and have cardinality $\binom{N-1}{2}$. Consider any flat connection with constant coefficients in $\{0,1\}$ with corresponding ϕ or corresponding subset X in Proposition 4.2 of cardinality $|X| \geq \binom{N-1}{2}$. Suppose there exists $i \in \{1, \cdots, N\}$ such that for all i', $(ii') \notin X$. But there are only $\binom{N-1}{2}$ such elements of $\mathcal{C}$ (those not containing i in the 2-cycle) so X cannot have cardinality greater than this, hence $|X| = \binom{N-1}{2}$ and $\phi = \phi_i$. Otherwise, we suppose that for all i there exists i' such that $(ii') \in X$. Then for any i, j we have $(ii'), (jj') \in X$ hence by the Ad closure of X we have $(ij) \in X$, i.e. $X = \mathcal{C}$ or $\phi = \theta$. $\diamond$

5. Metric structure

In this section we look at some more advanced aspects of the differential geometry for S_N, but for the 2-cycle calculus. First of all, just as the dual of the invariant 1-forms on a Lie group can be identified with the Lie algebra, the space $\mathcal{L} = \Lambda^{1*}$ for a bicovariant differential calculus on a coquasitriangular Hopf algebra A is typically a braided-Lie algebra in the sense introduced in [15]. Moreover, every braided-Lie algebra has an enveloping algebra[15] which in our case means

$$U(\mathcal{L}) = T\Lambda^{1*}/\mathrm{image}(\mathrm{id} - \Psi^*) = \Lambda^!_{quad}, \tag{12}$$

where ! is the quadratic algebra duality operation. There is also a canonical algebra homomorphism $U(\mathcal{L}) \to H$ where H is dual to A. We will call a differential structure 'connected' if this is a surjection. This theory applies to the Drinfeld-Jimbo $U_q(g)$ and gives it as generated by a braided-Lie algebra for each connected calculus.

However, the theory also applies to finite groups and in this case the axioms of a braided-Lie algebra reduce to what is called in algebraic topology a rack. Thus, given a conjugacy class on a finite group G, the associated rack or braided-Lie algebra is[2]

$$\mathcal{L} = \{x_a\}_{a\in\mathcal{C}}, \quad [x_a, x_b] = x_{b^{-1}ab}, \quad \Delta x_a = x_a \otimes x_a, \quad \epsilon(x_a) = 1. \tag{13}$$

The analogue of the Jacobi identity is

$$[[x_a, x_c], [x_b, x_c]] = [[x_a, x_b], x_c]. \tag{14}$$

The enveloping algebra is the ordinary bialgebra $U(\mathcal{L}) = k\langle x_a \rangle$ modulo the relations $x_a x_b = x_b x_{b^{-1}ab}$ and its homomorphism to the group algebra of G is $x_a \mapsto a$. This is surjective precisely when any element of G can be expressed as a product of elements of $\mathcal{C}$, i.e. by a path with respect to our differential structure (which determines the allowed steps as elements of $\mathcal{C}$) connecting the element to the group identity. Thus, in our finite group setting, the quadratic algebra Λ_{quad} is the !-dual of a fairly natural quadratic extension of the group algebra as an infinite-dimensional bialgebra. Note also that the flat connections in Proposition 4.2 define braided sub-Lie algebras.

Next, associated to any braided-Lie algebra is an Ad-invariant and braided-symmetric (with respect to Ψ) braided-Killing form, which may or may not be nondegenerate. This is computed in [2] for finite groups and one has

$$\eta^{a,b} \equiv \eta(x_a, x_b) = \#\{c \in \mathcal{C}|\ cab = abc\}. \tag{15}$$

The associated metric tensor in $\Omega^1 \otimes_{k(G)} \Omega^1$ is

$$\eta = \sum_{a,b} \eta^{a,b} e_a \otimes e_b$$

It is easy to see that among Ad-invariant η, 'braided symmetric' under Ψ is equivalent to symmetric in the usual sense. It is also equivalent (by definition of $\wedge$) to $\wedge(\eta) = 0$ under the exterior product.

Proposition 5.1. *For the braided-Lie algebra associated to the 2-cycle calculus on S_N, the braided-Killing form is*

$$\eta^{(ij),(ij)} = \binom{N}{2}, \quad \eta^{(ij),(km)} = \binom{N-4}{2} + 2, \quad \eta^{(ij),(jk)} = \binom{N-3}{2}$$

for distinct i, j, k, m. Moreover, the calculus is 'connected'.

Proof. All of $\mathcal{C}$ commutes with $(ij)^2 = e$. In the second case all elements disjoint from i, j, k, m and $(ij), (km)$ themselves commute with $(ij)(km)$. For the third case all elements disjoint from i, j, k commute with $(ij)(jk)$. The connectedness is the well-known property that the 2-cycles can be taken as generators of S_N. $\diamond$

To be a metric, we need η to be invertible, which we have verified explicitly at least up to $N < 30$. Other symmetric and invariant metrics also exist, not least $\eta^{a,b} = \delta_{a,b^{-1}}$, the Kronecker δ-function which is always invertible and works for any conjugacy class on any finite group that is stable under inversion. The general situation for S_N is:

Proposition 5.2. *The most general conjugation-invariant metric for the 2-cycle calculus on S_N has the symmetric form*

$$\eta^{(ij),(ij)} = \alpha, \quad \eta^{(ij),(km)} = \beta, \quad \eta^{(ij),(jk)} = \gamma$$

for distinct i, j, k, m, where α, β, γ are three arbitrary constants. Moreover,

$$\det(\eta) = (\alpha + \beta - 2\gamma)^{\frac{N(N-3)}{2}} (\alpha - (N-3)\beta + (N-4)\gamma)^{N-1} \cdot \left(\alpha + \frac{(N-2)(N-3)}{2}\beta + 2(N-2)\gamma\right)$$

at least up to $N \leq 10$.

Proof. Invariance here means $\eta^{gag^{-1},gbg^{-1}} = \eta^{a,b}$ for all $g \in G$. We use the mutually exclusive notations $a = b$, $a \perp b$ and $a \sim b$ as in the proof of Theorem 3.2, which is clearly an Ad-invariant decomposition of $\mathcal{C} \times \mathcal{C}$ (since the action of S_N is by a permutation of the 2-cycle entries). Clearly all the diagonal cases $a = b$ have the same value since $\mathcal{C}$ is a conjugacy class. Moreover, any $(ij) \perp (km)$ (for $N \geq 4$) is conjugate to $(12) \perp (34)$ by the choice of a suitable permutation (which we use to make the conjugation), so all of these have the same value. Similarly every $(ij) \sim (jk)$ (for $N \geq 3$) is conjugate to $(12) \sim (23)$, so these all have the same value. We then compute the determinants for $N \leq 10$ and find that they factorise in the form stated. The first two factors cancel in the case of $N = 2$. $\diamond$

Armed with an invertible metric, one may compute the associated Hodge-* operator, etc. as in [3] for S_3. The computation of this for S_N is beyond our present scope as it would require knowledge of Λ_{quad} in all degrees (we do not even know the dimensions for large N). It is also beyond our scope to recall all the details of noncommutative Riemannian geometry, but along the same lines as for S_3 in [2] we would expect a natural regular Levi-Civita connection with Ricci curvature tensor proportional to the metric modulo $\theta\otimes\theta$. Moreover, the same questions can be examined for the other conjugacy classes or 'Riemannian manifold' structures on S_N.

6. Braided group structure on $\mathcal{E}_N$

In this section we show that that the Fomin-Kirillov algebra $\mathcal{E}_N$ is a Hopf algebra in the braided category of crossed S_N-modules. In fact, we will find that like the exterior algebras Λ_N, it is a 'braided linear space' with additive coproduct on the generators[14]. We recall that a braided group B has a coproduct $\underline{\Delta}: B\to B\underline{\otimes}B$ which is coassociative and an algebra homomorphism provided the algebra $B\underline{\otimes}B$ is the braided-tensor product where

$$(a\otimes b)(c\otimes d)=a\Psi_{B,B}(b\otimes c)d$$

where $a,b,c,d\in B$ and $\Psi_{B,B}$ is the braiding on B. We show how the cross product (usual) Hopf algebras $kS_N{\triangleright\!\!\!<}\mathcal{E}_N$ in [5] and the skew derivations Δ_{ij} related to divided differences in [4] arise immediately as corollaries of the braided group structure. While the $\mathcal{E}_N$ are already-well studied by explicit means, we provide a more conceptual approach that is also more general and applies both to other conjugacy classes and to other groups beyond S_N.

As in [4] we consider that the algebra $\mathcal{E}_N$ is generated by an $[{N\atop 2}]$-dimensional vector space E_N (say) with basis $[ij]$ where $i<j$, and we extend the notation to $i>j$ by $[ij]=-[ji]$.

Theorem 6.1. *The algebras $\mathcal{E}_N$ are 'braided groups' or Hopf algebras in the category of S_N-crossed modules. Here*

$$g.[ij]=[g(i)\ g(j)]=\begin{cases}[g(i)\ g(j)] & \text{if } g(i)<g(j)\\ -[g(j)\ g(i)] & \text{if } g(i)>g(j)\end{cases},\ \forall g\in S_N,\quad |[ij]|=(ij)$$

is the crossed module structure on E_N, where $|\ |$ denotes the S_N-degree. Let Ψ denote the induced braiding, then

$$\mathcal{E}_N=TE_N/\ker(\mathrm{id}+\Psi),\quad \underline{\Delta}[ij]=[ij]\otimes 1+1\otimes[ij],\quad \underline{\epsilon}[ij]=0$$

is an additive braided group or 'linear braided space' in the category of S_N-crossed modules.

Proof. It is easy to verify that this is a crossed module structure. Thus $|g.[ij]|=\pm(g(i)\ g(j))=\pm g(ij)g^{-1}=|g[ij]g^{-1}|$ for the two cases (note that we consider the S_N-degree extended by linearity). The braiding is then

$$\Psi([ij]\otimes[km])=(ij).[km]\otimes[ij]$$

as defined by the crossed module structure. This is a signed version of the braiding used in Proposition 3.1 and by a similar analysis to the proof there, one finds that

the kernel of $\mathrm{id}+\Psi$ is precisely spanned by the relations of $\mathcal{E}_N$. In particular, note that

$$\Psi([ij]\otimes[ij]) = -[ij]\otimes[ij], \quad \Psi([ij]\otimes[km]) = [km]\otimes[ij]$$

if disjoint, which gives the relations $[ij][ij] = 0$ and $[ij][km] = [km][ij]$ when disjoint. Similarly for the 3-term relations $[ij][jk] + [jk][ki] + [ki][ij] = 0$ when i, j, k are distinct. Next, we define the coalgebra structure on the generators as stated and verify that these extend in a well-defined manner to a braided group structure on $\mathcal{E}_N$. there. This part is the same as for any braided-linear space [14] and we do not repeat it. The only presentational difference is that we directly define the relations as $\ker(\mathrm{id}+\Psi) = 0$ rather then seeking some other matrix Ψ' such that $\mathrm{image}(\mathrm{id}-\Psi') = \ker(\mathrm{id}+\Psi)$. $\diamond$

Corollary 6.2. *If $\mathcal{E}_N$ is finite-dimensional then it has a unique element in top degree.*

Proof. A top degree element would be an integral in the braided-Hopf algebra. But as for a usual Hopf algebra, the integral if it exists is unique (a formal proof in the braided case is in [18][19]). $\diamond$

Also, the biproduct bosonisation of any braided group B in the category of left A-crossed modules is an ordinary Hopf algebra $B{>\!\!\!\triangleleft\!\!\cdot} A$ (where A is an ordinary Hopf algebra with bijective antipode). This is the simultaneous cross product and cross coproduct in the construction of [16], in the braided group formulation [17, Appendix]. In our case A is finite dimensional so B also lives in the category of right A^*-crossed modules. Hence we immediately have two ordinary Hopf algebras, the first of which recovers the cross product observed in [5] and studied further there.

Corollary 6.3. *Biproduct bosonisation in the category of left crossed S_N-module structure gives an ordinary Hopf algebra $\mathcal{E}_N{>\!\!\!\triangleleft\!\!\cdot} kS_N$ with*

$$g[ij] = [g(i)\,g(j)]g, \quad \forall g\in S_N, \quad \Delta[ij] = [ij]\otimes 1 + (ij)\otimes[ij], \quad \epsilon[ij] = 0$$

extending that of kS_N, as in [5]*. Bosonisation in the equivalent category of right $k(S_N)$-crossed modules gives an ordinary Hopf algebra $k(S_N){\cdot\!\!\triangleright\!\!\!<}\mathcal{E}_N$ with*

$$[ij]f = R_{(ij)}(f)[ij], \quad \forall f\in k(S_N), \quad \Delta[ij] = \sum_{g\in S_N}[g(i)\,g(j)]\otimes\delta_g + 1\otimes[ij].$$

Proof. The kS_N-module structure defines the cross product and the kS_N-coaction $\Delta_L[ij] = (ij)\otimes[ij]$ defines the cross coproduct. In the second case the S_N-grading defines an action of $k(S_N)$ and the kS_N-module structure defines the $k(S_N)$-coaction $\Delta_R[ij] = \sum_g [g(i)\,g(j)]\otimes(ij)$ by dualisation. $\diamond$

Next, from a geometrical point of view the $\mathcal{E}_N$ are 'linear braided spaces', i.e. the coproduct $\underline{\Delta}$ corresponds to the additive group law on usual affine space in terms of its usual commutative polynomial algebra in several variables, but now in a braided-commutative version. We will use several results from this theory of linear braided spaces. For clarity we explicitly label the generators of $\mathcal{E}_N$ by 2-cycles. Thus $[ij] = e_{(ij)}$ if $i<j$. The products are different from those of Λ_N but we identify the basis of generators. In this notation we have

$$g.e_b = \zeta_{g,b}e_{gbg^{-1}}, \quad |e_a| = a \tag{16}$$

$$\zeta_{(ij),(ij)} = -1, \quad \zeta_{(ij),(km)} = 1, \quad \zeta_{(ij),(jk)} = \begin{cases} 1 & \text{if } i<j<k \\ 1 & \text{if } j<i<k \\ -1 & \text{if } j<k<i \\ -1 & \text{if } i<k<j \\ 1 & \text{if } k<i<j \\ 1 & \text{if } k<j<i \end{cases} \tag{17}$$

for i, j, k, m distinct, where ζ extends to S_N in its first argument by $\zeta_{gh,b} = \zeta_{g,hbh^{-1}}\zeta_{h,b}$ for all $g, h \in S_N$, i.e.,

$$\zeta \in Z^1_{\mathrm{Ad}}(S_N, k(\mathcal{C})); \quad \zeta(g)(b) = \zeta_{g,b},$$

as a multiplicative cocycle (using the multiplication of the algebra $k(\mathcal{C})$ of functions on $\mathcal{C}$ and with Ad the right action on $k(\mathcal{C})$ induced by conjugation). Thus, the algebras $\mathcal{E}_N$ differ from the exterior algebras Λ_N precisely by the introduction of a cocycle. This makes precise how to construct analogues of the $\mathcal{E}_N$ for other finite groups.

In this notation we have for any braided linear space[12][14]

$$\underline{\Delta}(e_{b_1}\cdots e_{b_m}) = \sum_{r=1}^{m} e_{c_1}\cdots e_{c_r} \otimes e_{c_{r+1}}\cdots e_{c_m} \begin{bmatrix} m \\ r \end{bmatrix}, \Psi\Big]^{c_1\cdots c_m}_{b_1\cdots b_m} \tag{18}$$

on products of generators. We view the braiding Ψ as a matrix (denoted PR in[14]). The braided binomial matrices have been introduced by the author in exactly this context and are not assumed to be invertible. There is also a braided antipode $\underline{S} : \mathcal{E}_N \to \mathcal{E}_N$ defined as -1 on the generators and extended braided-antimultiplicatively in the sense $\underline{S}(fg) = \cdot\Psi(\underline{S}f \otimes \underline{S}g)$ for all $f, g \in \mathcal{E}_N$, see [14]. In our case this comes out inductively as

$$\underline{S}(e_a f) = -\cdot\Psi(e_a \otimes \underline{S}f) = -(a.\underline{S}f)e_a, \quad \forall f \in \mathcal{E}_N. \tag{19}$$

Next we note that the braided group $\mathcal{E}_N$ is certainly finite-dimensional in each degree, so it has a graded-dual braided group $\mathcal{E}_N^*$. However, a linear braided space and its dual can typically be identified in the presence of an invariant metric. In our case we use the Kronecker $\delta_{a,b}$ metric and have:

Proposition 6.4. *$\mathcal{E}_N$ is self-dually paired as a braided group, with pairing*

$$\langle e_{a_n}\cdots e_{a_1}, e_{b_1}\cdots e_{b_m}\rangle = \delta_{n,m}([n,\Psi]!)^{a_1\cdots a_n}_{b_1\cdots b_n}$$

Proof. The pairing we take on the generators is $\langle e_a, e_b\rangle = \delta_{a,b}$, which is compatible with the S_N-grading since all a have order 2, and compatible with the action of $g \in S_N$ since $(\pm 1)^2 = 1$, i.e. the pairing is a morphism to the trivial crossed module. We extend this to products via the axioms of a braided group as explained in [14] to obtain the pairing stated, using the above formula for $\underline{\Delta}$ and properties of the braided binomial operators in relation to braided factorial matrices. It follows from the construction that the pairing is well-defined in its second input. This part is the same as in [14]. It is also well-defined in its first input after we observe that $\Psi^* = \Psi$, where Ψ^* is defined as the adjoint on $E_N \otimes E_N$ with respect to the braided-tensor pairing $E_N \otimes E_N \otimes E_N \otimes E_N \to k$ (in which we apply $\langle\ ,\ \rangle$ to the inner $E_N \otimes E_N$ first and then the outer two.) $\diamond$

This implies in particular that the two Hopf algebras $k(S_N){\triangleright\!\!\!<}\mathcal{E}_N$ and $\mathcal{E}_N{>\!\!\!\triangleleft} kS_N$ in Corollary 6.3 are dually paired. There are many more applications of the braided-linear space structure. As a less obvious one we compute the braided-Fourier theory [20] introduced for q-analysis on braided spaces.

Proposition 6.5. *For $\mathcal{E}_3$ the coevaluation for the pairing in Proposition 6.4 is*

$$\begin{aligned}\exp = {} & 1\otimes 1+[12]\otimes[12]+[23]\otimes[23]+[31]\otimes[31]\\ & -[12][23]\otimes[12][31]+[23][12]\otimes[12][23]+[23][31]\otimes[31][23]-[31][23]\otimes[31][12]\\ & +[31][12][23]\otimes[31][12][23]+[12][23][31]\otimes[12][23][31]+[23][31][12]\otimes[23][31][12]\\ & +[12][23][12][31]\otimes[12][23][12][31]\end{aligned}$$

and this along with the integration $\int$ defined as the coefficient of the top element $[12][23][12][31]$ *(and zero in lower degree) defines braided Fourier transform $\mathcal{S}$: $\mathcal{E}_3\to\mathcal{E}_3$*

$$\begin{aligned}&\mathcal{S}(1)=[12][23][12][31],\quad \mathcal{S}([12])=[31][12][23],\quad \mathcal{S}([23])=[12][23][31]\\ &\mathcal{S}([31])=[23][31][12],\quad \mathcal{S}([12][23])=[31][12],\quad \mathcal{S}([23][12])=[31][23]\\ &\mathcal{S}([23][31])=[12][23],\quad \mathcal{S}([31][23])=[12][31],\quad \mathcal{S}([31][12][23])=-[12]\\ &\mathcal{S}([12][23][31])=-[23],\quad \mathcal{S}([23][31][12])=-[31],\quad \mathcal{S}([12][23][12][31])=1.\end{aligned}$$

It obeys $\mathcal{S}^2=\mathrm{id}$ in degrees 0,4, $\mathcal{S}^2=-\mathrm{id}$ in degrees 1,3 and $\mathcal{S}^3=\mathrm{id}$ in degree 2.

Proof. Let $\{e^{(r)}_A\}$ be a basis of $\mathcal{E}_N$ in degree r and $\{f^{(r)A}\}$ the dual basis with respect to the pairing. In the nondegenerate case this is given by the inverse of the quotient operator $[r,\Psi]!$ acting on the degree r component of $\mathcal{E}_N$. The coevaluation for the pairing is

$$\exp=\sum_r\sum_A e^{(r)}_A\otimes f^{(r)A}$$

which computes as stated for $\mathcal{E}_3$. We take basis $\{[12],[23],[31]\}$ for degree 1, which is orthonormal with respect to the duality pairing. For degree 2 we take basis $\{[12][23],[23][12],[23][31],[31][23]\}$ with dual $\{-[12][31],[12][23],[31][23],-[31][12]\}$. The basis in degree 3 is the Fourier transform of the basis in degree 1 and orthonormal. The braided Fourier transform is defined on general braided groups possessing duals and integrals [20]. Here

$$\mathcal{S}(f)=(\int\otimes\mathrm{id})f\exp,\quad \forall f\in\mathcal{E}_N$$

which computes as stated using the relations of $\mathcal{E}_3$. A similar formula including the braided antipode $\underline{S}$ provides the inverse Fourier transform. We compute $\mathcal{S}^2$ as stated. In fact $\mathcal{S}^2=\mathcal{T}$, where $\mathcal{T}(f)=|f|.f$ on $f\in\mathcal{E}_N$ of homogeneous S_N-degree. One also has $\mathcal{S}=\mathcal{T}^{-1}=-\underline{S}$ in degree 2. $\diamond$

Also from the linear braided space structure, $\mathcal{E}_N^*$ and in our case $\mathcal{E}_N$ acts on the algebra $\mathcal{E}_N$ by infinitesimal translation from the left and right, which means respectively partial derivatives $D_a,\bar{D}_a:\mathcal{E}_N\to\mathcal{E}_N$ for each $a\in\mathcal{C}$ (they are denoted $\partial^a,\bar\partial^a$ in the general theory of [14]). Thus the left partial derivative is defined as the coefficient of $e_a\otimes$ in the operator $\underline{\Delta}$, which from the above yields

$$D_a(e_{a_1}\cdots e_{a_m})=e_{b_2}\cdots e_{b_m}[m,\Psi]^{ab_2\cdots b_m}_{a_1\cdots a_m}. \tag{20}$$

As for any braided linear space these necessarily represent $\mathcal{E}_N$ on itself and obey

$$D_a(fg) = D_a(f)g + \cdot\Psi^{-1}(D_a \otimes f)g, \quad \forall f, g \in \mathcal{E}_N, \tag{21}$$

making $\mathcal{E}_N$ an opposite braided $\mathcal{E}_N$-module algebra (i.e., in the braided category with inverted braiding). Similarly the right partial derivatives $\bar{D}_a$ are defined via right translations but converted into an action from the left via the braiding. This yields

$$\bar{D}_a(e_{a_1}\cdots e_{a_m}) = e_{b_2}\cdots e_{b_m}[m, \Psi^{-1}]^{ab_2\cdots b_m}_{a_1\cdots a_m} \tag{22}$$

and necessarily represent $\mathcal{E}_N$ on itself with

$$\bar{D}_a(fg) = \bar{D}_a(f)g + \cdot\Psi(\bar{D}_a \otimes f)g, \quad \forall f, g \in \mathcal{E}_N, \tag{23}$$

making $\mathcal{E}_N$ a $\mathcal{E}_N$-module algebra in its original braided category. The partial derivatives and their conjugates are related by the braided antipode according to

$$\underline{S}D_a = -\bar{D}_a\underline{S} \tag{24}$$

as shown in [21]. Proofs of all of these facts are by braid-diagram methods as part of our established theory of braided groups.

Corollary 6.6. *In the case of $\mathcal{E}_N$ the braided partial derivatives $D_a, \bar{D}_a$ are covariant (morphisms in the category of crossed modules) in the sense*

$$|D_a f| = a|f|, \quad g.D_a(f) = \zeta_{g,a} D_{gag^{-1}}(g.f), \quad \forall g \in S_N,\ f \in \mathcal{E}_N$$

(and similarly for $\bar{D}_a$). They obey $D_a(e_b) = \bar{D}_a(e_b) = \delta_{a,b}$ and the braided Leibniz rules

$$D_a(fg) = D_a(f)g + f\zeta_{|f|^{-1},a} D_{|f|^{-1}a|f|}(g), \quad \bar{D}_a(fg) = \bar{D}_a(f)g + (a.f)\bar{D}_a(g)$$

for all $f, g \in \mathcal{E}_N$ and f of homogeneous S_N-degree $|f|$ in the first case.

Proof. Whereas the above review of $D_a, \bar{D}_a$ holds for any additive braided group as part of a general theory, we specialize now to the particular braided category for the case of $\mathcal{E}_N$. First of all, the $D_a, \bar{D}_a$ are defined above as evaluation on e_a of morphisms $D, \bar{D} : E_N \otimes \mathcal{E}_N \to \mathcal{E}_N$. Thus $D(g.e_a \otimes g.f) = D(\zeta_{g,a} e_{gag^{-1}} \otimes g.f) = g.D(e_a \otimes f)$ translates to the condition as shown for $g \in S_N$ and $f \in \mathcal{E}_N$. Likewise, commuting with the total S_N-degree gives the other part of the morphism condition. Next we compute the braiding and its inverse on $E_N \otimes \mathcal{E}_N$ in the category of left crossed modules for the crossed module structure stated. In general $\Psi(f \otimes g) = |f|.g \otimes f$ for f of homogeneous S_N-degree $|f|$. Applying $D, \bar{D}$ yields the Leibniz rules as stated with

$$\Psi^{-1}(D_a \otimes f) = \zeta_{|f|^{-1},a}\, f \otimes D_{|f|^{-1}a|f|}, \quad \Psi(\bar{D}_a \otimes f) = a.f \otimes \bar{D}_a.$$

The braiding of D_a in these expressions corresponds by definition to the braiding of the element e_a which it represents (similarly for $\bar{D}_a$). $\diamond$

These $\bar{D}_{(ij)}$ therefore coincide with the operators denoted Δ_{ij} in the notation of [4]. It is proven there that their restriction to polynomials in the $\{\theta_i\}$ (i.e. to the cohomology of the flag variety) yields the finite difference operators

$$\partial_{ij} f = \frac{f - (ij).f}{\theta_i - \theta_j} \tag{25}$$

where $(ij).f$ interchanges the i, j arguments of $f(\theta_1, \cdots, \theta_N)$. We see that these $\bar{D}_a$ follow directly from the braided group structure as infinitesimal translations,

which ensures that they are well-defined and form a representation of $\mathcal{E}_N$ on itself. It also provides computational tools, for example braided-Fourier transform intertwines the braided derivatives with multiplication in $\mathcal{E}_N$, as shown in general in [20]. Another canonically-defined representation of a braided group on itself with a similar braided-Leibniz property to the $\bar{D}_a$ is the braided adjoint action, which comes out for $\mathcal{E}_N$ as

$$\underline{\mathrm{Ad}}_{e_a}(f) \equiv e_a f + \cdot\Psi(\underline{S}e_a \otimes f) = e_a f - (a.f)e_a \tag{26}$$

$$\underline{\mathrm{Ad}}_{e_a}(fg) = \underline{\mathrm{Ad}}_{e_a}(f)g + (a.f)\underline{\mathrm{Ad}}_{e_a}(g). \tag{27}$$

This has no direct geometrical analogue (the usual polynomial algebra is commutative so that $\underline{\mathrm{Ad}}$ is zero).

Let us also note that the full bosonisation theorem[14, Thm. 9.4.12] of $\mathcal{E}_N$ provides another ordinary Hopf algebra $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot} D(kS_N)$ such that its category of modules is fully equivalent to the category of braided modules of the braided group $\mathcal{E}_N$. In particular, module-algebras of this ordinary Hopf algebra are the same thing as braided $\mathcal{E}_N$-module algebras, such as provided by $\bar{D}_a, \mathrm{Ad}_{e_a}$ above. The Drinfeld double of a finite group is itself a semidirect product $D(kS_N) = k(S_N)_{\mathrm{Ad}}{>\!\!\!\triangleleft} kS_N$.

Corollary 6.7. *The full bosonisation Hopf algebra $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot}(k(S_N){>\!\!\!\triangleleft} kS_N)$ contains $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot} kS_N$ in Corollary 6.3 and $k(S_N)$ as sub-Hopf algebras with additional relations*

$$f[ij] = [ij]\, L_{(ij)}(f), \quad gf = \mathrm{Ad}_{g^{-1}}(f)g, \quad \forall g \in S_N,\ f \in k(S_N)$$

where $L_g(f) = f(g(\ ))$ and $\mathrm{Ad}_g = L_g R_{g^{-1}}$. The same algebra has another 'conjugate' coproduct containing $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot} k(S_N)$ and kS_N as sub-Hopf algebras.

Proof. The right action of $k(S_N)$ on $\mathcal{E}_N$ given by the grading can also be used as a left action. This action and the action of kS_N is the action of the Drinfeld double corresponding to the crossed module. We make the semidirect product by this. The quasitriangular structure $\mathcal{R} = \sum_{g\in S_N} \delta_g \otimes g$ defines a left coaction $\Delta_L(e_a) = \mathcal{R}_{21}.e_a = \sum_{g\in S_N} g \otimes \delta_g.e_a = a \otimes e_a$ induced from the action, so the same cross coproduct as for $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot} kS_N$. On the other hand every quasitriangular Hopf algebra has a conjugate quasitriangular structure $\bar{\mathcal{R}} = \mathcal{R}_{21}^{-1}$. We regard the same algebra $\mathcal{E}_N$ developed as a braided group in this opposite braided category (the opposite braided coproduct looks the same on the generators E_N.) Using $\bar{\mathcal{R}}$ gives a second induced coaction $\bar{\Delta}_L(e_a) = \sum_{g\in S_N} \delta_{g^{-1}} \otimes g.e_a$. This gives a second ordinary coproduct

$$\bar{\Delta}[ij] = [ij] \otimes 1 + \sum_{g\in S_N} \delta_{g^{-1}} \otimes [g(i), g(j)]$$

which is a left handed version $\mathcal{E}_N{>\!\!\!\triangleleft\kern-.33em\cdot} k(S_N)$ of the second biproduct bosonisation in Corollary 6.3. This is an example of a general theory in [21] where the two coproducts are related by complex conjugation in a $*$-algebra setting over $\mathbb{C}$. $\diamond$

Having understood the structure of $\mathcal{E}_N$ in a natural way, let us note now that all of the above applies equally well to the full quotient of it

$$\mathcal{E}_w = TE_N / \oplus_n \ker Sym_n, \quad Sym_n = \sum_{\sigma = s_{i_1}\cdots s_{i_{l(\sigma)}} \in S_n} \Psi_{i_1} \cdots \Psi_{i_{l(\sigma)}} \tag{28}$$

where in principle there could be nonquadratic relations. In this case, since $Sym_n = [n,\Psi]!$, it is clear that here the pairings are now nondegenerate (we have divided by the coradicals of the pairing in Proposition 6.4). Therefore $\mathcal{E}_w$ is a self-dual braided group. If finite-dimensional then it would inherit a symmetric Hilbert series as explained in Section 3. Also for the reasons given there, we expect that $\mathcal{E}_N$ and $\mathcal{E}_w$ coincide and the latter if finite dimensional will have a symmetric Hilbert series which will prove the conjecture of a symmetric Hilbert series for $\mathcal{E}_N$ made in [4]. But if they do not coincide, we propose $\mathcal{E}_w$ as the better-behaved version of $\mathcal{E}_N$; it may be that $\mathcal{E}_w$ is finite-dimensional while the $\mathcal{E}_N$ is likely not to be for $N \geq 6$. Thus we propose a potential and better behaved quotient of $\mathcal{E}_N$. Moreover, our braided group methods work for general finite groups where we would not expect $\mathcal{E}_w$ to be quadratic and which would probably be needed for flag varieties associated to different Lie algebras beyond SL_N. This is a proposal for further work.

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Appendix A. Braided group structure of Λ_w

Here we will say a little more about the general theory behind exterior algebras Λ_{quad} or Λ_w than covered in the Preliminaries in Section 2. This is needed for some of the remarks about Hodge * operator mentioned in Sections 3,5 and is also the motivation behind the results given directly for $\mathcal{E}_N$ in Section 6. It was considered too technical to be put in the main text.

First of all, the Woronowicz construction Ω_w on a quantum group A is usually given as a quotient of the tensor algebra on Ω^1 over A. We have instead moved everything over to the left-invariant forms Λ_w which is a 'braided approach' to the exterior algebra in [9][11]. See also [10]. The starting point is that associated to any linear space Λ^1 equipped with a Yang-Baxter or braid operator (in our case $-\Psi$) one has Λ_{quad} (and similarly Λ_w) braided linear spaces with additive coproduct

$$\underline{\Delta} e_a = e_a \otimes 1 + 1 \otimes e_a, \quad \underline{\epsilon} e_a = 0. \tag{29}$$

In our case these live in the braided category which is a $\mathbb{Z}_2$ extension of the category of A-crossed modules, with Λ^1 odd. Thus one may verify:

$$\underline{\Delta}(e_a e_b) = (e_a \otimes 1 + 1 \otimes e_a)(e_b \otimes 1 + 1 \otimes e_b) = e_a e_b \otimes 1 + 1 \otimes e_a e_b + (\mathrm{id} - \Psi)(e_a \otimes e_b).$$

If $\lambda_{a,b} e_a e_b = 0$ (summation understood) then $\underline{\Delta}$ of it is also zero since the relation in degree 2 is exactly that $(\mathrm{id} - \Psi)(\lambda_{a,b} e_a \otimes e_b) = 0$. This covers Λ_{quad}. For Λ_w one has to similarly look at the higher degrees. Similarly to Section 6 there is

then a super-biproduct bosonisation theorem which yields Ω_{quad} and Ω_w as super-Hopf algebras by crossed module constructions. We also have super-braided-partial derivatives $D_a, \bar{D}_a$, which define interior products[9].

Here we would like to say a little more as an explanation of the definition of Λ_w. Let Λ^{1*} be the crossed module with adjoint braiding Ψ^*. It has its own algebra of 'skew invariant tensor fields'

$$\Lambda^*_{quad} = T\Lambda^{1*}/\ker(\mathrm{id} - \Psi^*), \quad \Lambda^*_w = T\Lambda^{1*}/\oplus_n A^*_n. \tag{30}$$

Proposition A.1. *The tensor algebras $T\Lambda$ and $T\Lambda^{1*}$ are dually paired braided groups as induced by the pairing in degree* 1*, and $\Lambda_w, \Lambda^{1*}_w$ are their quotients by the kernel of the pairing.*

Proof. Let $\{f^a\}$ be the dual basis of Λ^{1*}. The pairing between monomials in the tensor algebra is then

$$\langle f^{a_n}\cdots f^{a_1}, e_{b_1}\cdots e_{b_m}\rangle = \delta_{n,m}[n, -\Psi]!^{a_1\cdots a_n}_{b_1\cdots b_n}$$

as for any braided linear space [14]. In view of Proposition 2.1 we are therefore defining Λ_w exactly by killing the kernel of the pairing from that side. Similarly from the other side. ⋄

This the meaning of the Woronowicz construction is that one adds enough relations that the pairing with its similar dual version is non-degenerate. Moreover, as in Section 6, we know that by the theory of integrals on braided groups, if Λ_w is finite dimensional then there is a unique top form Top, of degree d say. In this case there is an approach to a Hodge * pairing in [9] based on braided-differentiation of the top form and related to braided Fourier transform. A similar and more explicit version of this which has been used in [3] to define an 'epsilon tensor' by $e_{a_1}\cdots e_{a_d} = \epsilon_{a_1\cdots a_d}\mathrm{Top}$ and then use this to define a map $\Lambda^m_w \to \Lambda^{*(d-m)}_w$. In the presence of an invariant metric we have $\Lambda^1 \cong \Lambda^{*1}$ as crossed modules and hence isomorphisms of their generated braided groups. In this case we have a Hodge * operator $\Lambda^m_w \to \Lambda^{d-m}_w$. Similarly if Λ_{quad} is finite dimensional.

In the case of a finite group G with calculus defined by a conjugacy class $\mathcal{C}$, we compute

$$\Psi^*(f^a \otimes f^b) = f^{a^{-1}ba} \otimes f^a \tag{31}$$

where the adjoint is taken with respect to the pairing on tensor powers (recall that conventionally this is defined by pairing the inner factors first and moving outwards, to avoid unnecessary braid crossings). We let Λ denote either Λ_{quad} or Λ_w (or an intermediate quotient).

Corollary A.2. *If $\mathcal{C}$ is stable under group inversion then Λ is self-dually paired as a braided group. If Λ is finite-dimensional with top degree d we have*

$$*(e_{a_1}\cdots e_{a_m}) == d_m^{-1}\epsilon_{a_1\cdots a_d} e_{a_d^{-1}}\cdots e_{a_{m+1}^{-1}} \tag{32}$$

for some normalisations d_m.

Proof. In this case we have an invariant metric $\eta^{a,b} = \delta_{a,b^{-1}}$ whereby we identify $f^a = e_{a^{-1}}$. For Λ_w in the algebraically closed case one would typically chose the d_m so that $*^2 = \mathrm{id}$. The formula as in [3] is arranged to be covariant so that if Top is invariant under the $k(G)$-action (which implies that it commutes with functions) then $*$ will extend to a bimodule map $\Omega^m \to \Omega^{d-m}$. ⋄

Similarly, the exterior algebra Ω is generated in the finite group case by $k(G)$ and Λ with the cross relations (5), which is manifestly a cross product $k(G)\triangleright\!\!<\Lambda$. The super coalgebra explicitly is

$$\Delta e_a=\sum_{g\in G}e_{gag^{-1}}\otimes\delta_g+1\otimes e_a,\quad \epsilon e_a=0 \tag{33}$$

and extends the group coordinate Hopf algebra. Here δ_g is a delta-function on G. Indeed, the G-grading part of the crossed module structure on Λ^1 extends to all of Λ and defines a right action of $k(G)$ on it (by evaluating against the total G-degree) which is used in the cross product algebra. Meanwhile, the left G-action defines a right coaction of $k(G)$,

$$\Delta_R(e_a)=\sum_g e_{gag^{-1}}\otimes\delta_g \tag{34}$$

which extends as an algebra homomorphism to Λ because Ψ is Ad-covariant. Semidirect coproduct by this defines the coalgebra of $k(G)\triangleright\!\!<\Lambda$. The two fit together to form a super-Hopf algebra just because the original structure on Λ^1 was a crossed module. For example, one may check

$$\begin{aligned}\Delta(e_ae_b)&=(\sum_g e_{gag^{-1}}\otimes\delta_g+1\otimes e_a)(\sum_h e_{hbh^{-1}}\otimes\delta_h+1\otimes e_b)\\&=1\otimes e_ae_b+\sum_g e_{gag^{-1}}e_{gbg^{-1}}\otimes\delta_g+e_{gag^{-1}}\otimes\delta_ge_b-e_{gbg^{-1}}\otimes e_a\delta_g\\&=(1\otimes\cdot+\Delta_R\circ\cdot+(\Delta_R\otimes\mathrm{id})(\mathrm{id}-\Psi))\,e_a\otimes e_b\end{aligned}$$

since we are extending as a super-Hopf algebra (so Λ^1 is odd). We used the relations in the algebra and changed variables in the last term. From this it is clear that Δ is well defined in the quotient by $\ker(\mathrm{id}-\Psi)$. This covers $\Lambda=\Lambda_{quad}$ but the same holds also for Λ_w.

Similarly, since a right $k(G)$-crossed module is the same thing as a left kG-crossed module, we can make another super-Hopf algebra $\Lambda{>\!\!\triangleleft} kG$. We extend the G-action $g.e_a=e_{gag^{-1}}$ to Λ for the cross product and the grading defines a left coaction

$$\Delta_Le_a=a\otimes e_a \tag{35}$$

which we extend to products (expressing the total G-degree). Semidirect product and coproduct by these gives

$$ge_a=e_{gag^{-1}}g,\quad \Delta e_a=e_a\otimes 1+a\otimes e_a,\quad \epsilon e_a=0 \tag{36}$$

extending the Hopf algebra structure of the group algebra kG. This time

$$\begin{aligned}\Delta(e_ae_b)&=e_ae_b\otimes 1+ab\otimes e_ae_b+e_ab\otimes e_b-ae_b\otimes e_a\\&=(\cdot\otimes 1)+\Delta_L\circ\cdot+(\mathrm{id}\otimes\Delta_L)(\mathrm{id}-\Psi))\,e_a\otimes e_b\end{aligned}$$

which is well-defined on the quotient . Geometrically, this is the dual of the super-Hopf algebra $k(G)\triangleright\!\!<\Lambda^*$ of skew-vector fields. These are the direct contructions of the cross products analogous to those in Section 6 for $\mathcal{E}_N$. We have similar pairing results.

Finally, let us note that at this level of generality all the same proofs work with $-\Psi$ replaced by Ψ. Thus for any crossed module E with braiding Ψ we have a braided space

$$\mathcal{E}_{quad}=TE/\ker(\mathrm{id}+\Psi) \tag{37}$$

and similarly $\mathcal{E}_w$ defined by Sym_n as in (28), both forming additive braided groups. Moreover, one should be able to construct a suitable crossed module from any conjugacy class on a finite group and possibly a cocycle ζ. This indicates how the analogues of the Fomin-Kirillov algebra could be extended to other types.

References

[1] S.L. Woronowicz. Differential calculus on compact matrix pseudogroups (quantum groups). *Commun. Math. Phys.*, 122:125–170, 1989.

[2] S. Majid. Riemannian geometry of quantum groups and finite groups with nonuniversal differentials. *Commun. Math. Phys.*, 225:131-170, 2002.

[3] S. Majid and E. Raineri. Electromagnetism and gauge theory on the permutation group S_3. *J. Geom. Phys.*, 44:129-155, 2002.

[4] S. Fomin and A.N. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. *Adv. Geom, Progr. Math.*, 172:147–182. Birkhauser, 1999.

[5] S. Fomin and C. Procesi. Fibered quadratic Hopf algebras related to Schubert calculus. *J. Algebra*, 230:174–183, 2000.

[6] F.R. Cohen, T.J. Lada and J.P. May. The homology of iterated loop spaces. *Springer Lecture Notes in Math* 533, 1976.

[7] G.I. Lehrer. Equivariant cohomology of configurations in $\mathbb{R}^d$. *Algebras and Repn. Theory*, 3:377-384, 2000.

[8] A. Connes. *Noncommutative Geometry.* Academic Press, 1994.

[9] S. Majid. q-Epsilon tensor for quantum and braided spaces. *J. Math. Phys.*, 36:1991–2007, 1995.

[10] M. Rosso. Groupes quantiques et algebres de battage quantiques. *C.R.A.C.* 320:145–148, 1995.

[11] Yu. N. Bespalov and B. Drabant. Differential calculus in braided Abelian categories. *Preprint*, q-alg/9703036.

[12] S. Majid. Free braided differential calculus, braided binomial theorem and the braided exponential map. *J. Math. Phys.*, 34:4843–4856, 1993.

[13] K. Bresser, F. Mueller-Hoissen, A. Dimakis, A. Sitarz. Noncommutative geometry of finite groups. *J. Phys. A* 29:2705–2736, 1996.

[14] S. Majid. *Foundations of Quantum Group Theory.* Cambridge Univeristy Press, 1995.

[15] S. Majid. Quantum and braided Lie algebras. *J. Geom. Phys.*, 13:307–356, 1994.

[16] D. Radford. The structure of Hopf algebras with a projection. *J. Algebra* 92:322–347, 1985.

[17] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Comm. Math. Phys.* 156:607–638, 1993.

[18] V. Lyubashenko. Modular transformations for tensor categories. *J. Pure Appl. Alg.* 98:279–327, 1995.

[19] Y. Bespalov, T. Kerler, V. Lyubashenko and V. Turaev. Integrals for braided Hopf algebras. *J. Pure Appl. Alg.*, 148:113–164, 2000.

[20] A. Kempf and S. Majid. Algebraic q-integration and Fourier theory on quantum and braided Spaces. *J. Math. Phys.* 35: 6802–6837, 1994.

[21] S. Majid. Quasi-* structure on q-Poincaré algebras. *J. Geom. Phys.* 22:14–58, 1997.

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